

# NAG Toolbox for MATLAB

## e02da

### 1 Purpose

e02da forms a minimal, weighted least-squares bicubic spline surface fit with prescribed knots to a given set of data points.

### 2 Syntax

```
[lamda, mu, dl, c, sigma, rank, ifail] = e02da(x, y, f, w, lamda, mu,
point, eps, 'm', m, 'px', px, 'py', py, 'npoint', npoint)
```

### 3 Description

e02da determines a bicubic spline fit  $s(x,y)$  to the set of data points  $(x_r, y_r, f_r)$  with weights  $w_r$ , for  $r = 1, 2, \dots, m$ . The two sets of internal knots of the spline,  $\{\lambda\}$  and  $\{\mu\}$ , associated with the variables  $x$  and  $y$  respectively, are prescribed by you. These knots can be thought of as dividing the data region of the  $(x,y)$  plane into panels (see Figure 1 in Section 5). A bicubic spline consists of a separate bicubic polynomial in each panel, the polynomials joining together with continuity up to the second derivative across the panel boundaries.

$s(x,y)$  has the property that  $\Sigma$ , the sum of squares of its weighted residuals  $\rho_r$ , for  $r = 1, 2, \dots, m$ , where

$$\rho_r = w_r(s(x_r, y_r) - f_r) \quad (1)$$

is as small as possible for a bicubic spline with the given knot sets. The function produces this minimized value of  $\Sigma$  and the coefficients  $c_{ij}$  in the B-spline representation of  $s(x,y)$  – see Section 8. e02de and e02df are available to compute values of the fitted spline from the coefficients  $c_{ij}$ .

The least-squares criterion is not always sufficient to determine the bicubic spline uniquely: there may be a whole family of splines which have the same minimum sum of squares. In these cases, the function selects from this family the spline for which the sum of squares of the coefficients  $c_{ij}$  is smallest: in other words, the minimal least-squares solution. This choice, although arbitrary, reduces the risk of unwanted fluctuations in the spline fit. The method employed involves forming a system of  $m$  linear equations in the coefficients  $c_{ij}$  and then computing its least-squares solution, which will be the minimal least-squares solution when appropriate. The basis of the method is described in Hayes and Halliday 1974. The matrix of the equation is formed using a recurrence relation for B-splines which is numerically stable (see Cox 1972a and de Boor 1972 – the former contains the more elementary derivation but, unlike de Boor 1972, does not cover the case of coincident knots). The least-squares solution is also obtained in a stable manner by using orthogonal transformations, viz. a variant of Givens rotation (see Gentleman 1973). This requires only one row of the matrix to be stored at a time. Advantage is taken of the stepped-band structure which the matrix possesses when the data points are suitably ordered, there being at most sixteen nonzero elements in any row because of the definition of B-splines. First the matrix is reduced to upper triangular form and then the diagonal elements of this triangle are examined in turn. When an element is encountered whose square, divided by the mean squared weight, is less than a threshold  $\epsilon$ , it is replaced by zero and the rest of the elements in its row are reduced to zero by rotations with the remaining rows. The rank of the system is taken to be the number of nonzero diagonal elements in the final triangle, and the nonzero rows of this triangle are used to compute the minimal least-squares solution. If all the diagonal elements are nonzero, the rank is equal to the number of coefficients  $c_{ij}$  and the solution obtained is the ordinary least-squares solution, which is unique in this case.

### 4 References

- Cox M G 1972a The numerical evaluation of B-splines *J. Inst. Math. Appl.* **10** 134–149  
 de Boor C 1972 On calculating with B-splines *J. Approx. Theory* **6** 50–62

Gentleman W M 1973 Least-squares computations by Givens transformations without square roots *J. Inst. Math. Applic.* **12** 329–336

Hayes J G and Halliday J 1974 The least-squares fitting of cubic spline surfaces to general data sets *J. Inst. Math. Appl.* **14** 89–103

## 5 Parameters

### 5.1 Compulsory Input Parameters

- 1: **x(m)** – double array
- 2: **y(m)** – double array
- 3: **f(m)** – double array

The co-ordinates of the data point  $(x_r, y_r, f_r)$ , for  $r = 1, 2, \dots, m$ . The order of the data points is immaterial, but see the array **point**.

- 4: **w(m)** – double array

The weight  $w_r$  of the  $r$ th data point. It is important to note the definition of weight implied by the equation (1) in Section 3, since it is also common usage to define weight as the square of this weight. In this function, each  $w_r$  should be chosen inversely proportional to the (absolute) accuracy of the corresponding  $f_r$ , as expressed, for example, by the standard deviation or probable error of the  $f_r$ . When the  $f_r$  are all of the same accuracy, all the  $w_r$  may be set equal to 1.0.

- 5: **lamda(px)** – double array

**lamda**( $i+4$ ) must contain the  $i$ th interior knot  $\lambda_{i+4}$  associated with the variable  $x$ , for  $i = 1, 2, \dots, \mathbf{px} - 8$ . The knots must be in nondecreasing order and lie strictly within the range covered by the data values of  $x$ . A knot is a value of  $x$  at which the spline is allowed to be discontinuous in the third derivative with respect to  $x$ , though continuous up to the second derivative. This degree of continuity can be reduced, if you require, by the use of coincident knots, provided that no more than four knots are chosen to coincide at any point. Two, or three, coincident knots allow loss of continuity in, respectively, the second and first derivative with respect to  $x$  at the value of  $x$  at which they coincide. Four coincident knots split the spline surface into two independent parts. For choice of knots see Section 8.

- 6: **mu(py)** – double array

**mu**( $i+4$ ) must contain the  $i$ th interior knot  $\mu_{i+4}$  associated with the variable  $y$ , for  $i = 1, 2, \dots, \mathbf{py} - 8$ .

- 7: **point(npoint)** – int32 array

Indexing information usually provided by e02za which enables the data points to be accessed in the order which produces the advantageous matrix structure mentioned in Section 3. This order is such that, if the  $(x, y)$  plane is thought of as being divided into rectangular panels by the two sets of knots, all data in a panel occur before data in succeeding panels, where the panels are numbered from bottom to top and then left to right with the usual arrangement of axes, as indicated in Figure

1.

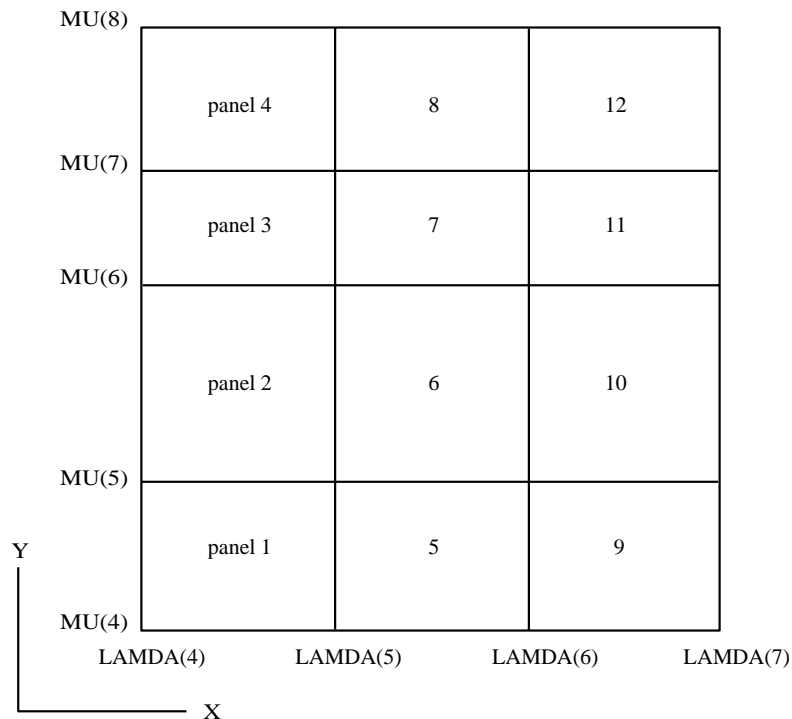


Figure 1

A data point lying exactly on one or more panel sides is considered to be in the highest numbered panel adjacent to the point. `e02za` should be called to obtain the array **point**, unless it is provided by other means.

#### 8: **eps – double scalar**

A threshold  $\epsilon$  for determining the effective rank of the system of linear equations. The rank is determined as the number of elements of the array **dl** which are nonzero. An element of **dl** is regarded as zero if it is less than  $\epsilon$ . *Machine precision* is a suitable value for  $\epsilon$  in most practical applications which have only 2 or 3 decimals accurate in data. If some coefficients of the fit prove to be very large compared with the data ordinates, this suggests that  $\epsilon$  should be increased so as to decrease the rank. The array **dl** will give a guide to appropriate values of  $\epsilon$  to achieve this, as well as to the choice of  $\epsilon$  in other cases where some experimentation may be needed to determine a value which leads to a satisfactory fit.

## 5.2 Optional Input Parameters

#### 1: **m – int32 scalar**

*Default:* The dimension of the arrays **x**, **y**, **f**, **w**. (An error is raised if these dimensions are not equal.)

*m*, the number of data points.

*Constraint:*  $m > 1$ .

#### 2: **px – int32 scalar**

#### 3: **py – int32 scalar**

*Default:* The dimension of the array **lamda**.

the total number of knots  $\lambda$  and  $\mu$  associated with the variables  $x$  and  $y$ , respectively.

*Constraint:*  $\mathbf{px} \geq 8$  and  $\mathbf{py} \geq 8$ .

(They are such that  $\mathbf{px} - 8$  and  $\mathbf{py} - 8$  are the corresponding numbers of interior knots.) The running time and storage required by the function are both minimized if the axes are labelled so that  $\mathbf{py}$  is the smaller of  $\mathbf{px}$  and  $\mathbf{py}$

4: **npoint – int32 scalar**

*Default:* The dimension of the array **point**.

*Constraint:*  $\mathbf{npoint} \geq \mathbf{m} + (\mathbf{px} - 7) \times (\mathbf{py} - 7)$ .

### 5.3 Input Parameters Omitted from the MATLAB Interface

nc, ws, nws

### 5.4 Output Parameters

1: **lamda(px) – double array**

The interior knots **lamda**(5) to **lamda**( $\mathbf{px} - 4$ ) are unchanged, and the segments **lamda**(1 : 4) and **lamda**( $\mathbf{px} - 3$  :  $\mathbf{px}$ ) contain additional (exterior) knots introduced by the function in order to define the full set of B-splines required. The four knots in the first segment are all set equal to the lowest data value of  $x$  and the other four additional knots are all set equal to the highest value: there is experimental evidence that coincident end-knots are best for numerical accuracy. The complete array must be left undisturbed if e02de or e02df is to be used subsequently.

2: **mu(py) – double array**

The same remarks apply to **mu** as to **lamda** above, with **y** replacing **x**, and  $y$  replacing  $x$ .

3: **dl(nc) – double array**

Gives the squares of the diagonal elements of the reduced triangular matrix, divided by the mean squared weight. It includes those elements, less than  $\epsilon$ , which are treated as zero (see Section 3).

4: **c(nc) – double array**

Gives the coefficients of the fit. **c**(( $\mathbf{py} - 4$ )  $\times$  ( $i - 1$ ) +  $j$ ) is the coefficient  $c_{ij}$  of Sections 3 and 8 for  $i = 1, 2, \dots, \mathbf{px} - 4$  and  $j = 1, 2, \dots, \mathbf{py} - 4$ . These coefficients are used by e02de or e02df to calculate values of the fitted function.

5: **sigma – double scalar**

$\Sigma$ , the weighted sum of squares of residuals. This is not computed from the individual residuals but from the right-hand sides of the orthogonally-transformed linear equations. For further details see page 97 of Hayes and Halliday 1974. The two methods of computation are theoretically equivalent, but the results may differ because of rounding error.

6: **rank – int32 scalar**

The rank of the system as determined by the value of the threshold  $\epsilon$ .

**rank = nc**

The least-squares solution is unique.

**rank  $\neq$  nc**

The minimal least-squares solution is computed.

7: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

## 6 Error Indicators and Warnings

Errors or warnings detected by the function:

**ifail** = 1

At least one set of knots is not in nondecreasing order, or an interior knot is outside the range of the data values.

**ifail** = 2

More than four knots coincide at a single point, possibly because all data points have the same value of  $x$  (or  $y$ ) or because an interior knot coincides with an extreme data value.

**ifail** = 3

Array **point** does not indicate the data points in panel order. Call e02za to obtain a correct array.

**ifail** = 4

On entry, **m**  $\leq 1$ ,  
 or **px**  $< 8$ ,  
 or **py**  $< 8$ ,  
 or **nc**  $\neq (\mathbf{px} - 4) \times (\mathbf{py} - 4)$ ,  
 or **nws** is too small,  
 or **npoint** is too small.

**ifail** = 5

All the weights  $w_r$  are zero or rank determined as zero.

## 7 Accuracy

The computation of the B-splines and reduction of the observation matrix to triangular form are both numerically stable.

## 8 Further Comments

The time taken is approximately proportional to the number of data points,  $m$ , and to  $(3 \times (\mathbf{py} - 4) + 4)^2$ .

The B-spline representation of the bicubic spline is

$$s(x, y) = \sum_{i,j} c_{ij} M_i(x) N_j(y)$$

summed over  $i = 1, 2, \dots, \mathbf{px} - 4$  and over  $j = 1, 2, \dots, \mathbf{py} - 4$ . Here  $M_i(x)$  and  $N_j(y)$  denote normalized cubic B-splines, the former defined on the knots  $\lambda_i, \lambda_{i+1}, \dots, \lambda_{i+4}$  and the latter on the knots  $\mu_j, \mu_{j+1}, \dots, \mu_{j+4}$ . For further details, see Hayes and Halliday 1974 for bicubic splines and de Boor 1972 for normalized B-splines.

The choice of the interior knots, which help to determine the spline's shape, must largely be a matter of trial and error. It is usually best to start with a small number of knots and, examining the fit at each stage, add a few knots at a time in places where the fit is particularly poor. In intervals of  $x$  or  $y$  where the surface represented by the data changes rapidly, in function value or derivatives, more knots will be needed than elsewhere. In some cases guidance can be obtained by analogy with the case of coincident knots: for example, just as three coincident knots can produce a discontinuity in slope, three close knots can produce rapid change in slope. Of course, such rapid changes in behaviour must be adequately represented by the data points, as indeed must the behaviour of the surface generally, if a satisfactory fit is to be achieved. When there is no rapid change in behaviour, equally-spaced knots will often suffice.

In all cases the fit should be examined graphically before it is accepted as satisfactory.

The fit obtained is not defined outside the rectangle

$$\lambda_4 \leq x \leq \lambda_{\mathbf{px}-3}, \mu_4 \leq y \leq \mu_{\mathbf{py}-3}.$$

The reason for taking the extreme data values of  $x$  and  $y$  for these four knots is that, as is usual in data fitting, the fit cannot be expected to give satisfactory values outside the data region. If, nevertheless, you require values over a larger rectangle, this can be achieved by augmenting the data with two artificial data points  $(a, c, 0)$  and  $(b, d, 0)$  with zero weight, where  $a \leq x \leq b$ ,  $c \leq y \leq d$  defines the enlarged rectangle. In the case when the data are adequate to make the least-squares solution unique (**rank** = **nc**), this enlargement will not affect the fit over the original rectangle, except for possibly enlarged rounding errors, and will simply continue the bicubic polynomials in the panels bordering the rectangle out to the new boundaries: in other cases the fit will be affected. Even using the original rectangle there may be regions within it, particularly at its corners, which lie outside the data region and where, therefore, the fit will be unreliable. For example, if there is no data point in panel 1 of Figure 1 in Section 5, the least-squares criterion leaves the spline indeterminate in this panel: the minimal spline determined by the (sub)program in this case passes through the value zero at the point  $(\lambda_4, \mu_4)$ .

## 9 Example

```
x = [0.6;
     -0.95;
      0.87;
      0.84;
      0.17;
     -0.87;
      1;
      0.1;
      0.24;
     -0.77;
      0.32;
      1;
     -0.63;
     -0.66;
      0.93;
      0.15;
      0.99;
     -0.54;
      0.44;
     -0.72;
      0.63;
     -0.4;
      0.2;
      0.43;
      0.28;
     -0.24;
      0.86;
     -0.41;
     -0.05;
     -1];
y = [-0.52;
     -0.61;
      0.93;
      0.09;
      0.88;
     -0.7;
      1;
      1;
      0.3;
     -0.77;
     -0.23;
     -1;
     -0.26;
     -0.83;
      0.22;
      0.89;
```

```
-0.8;
-0.88;
0.68;
-0.14;
0.67;
-0.9;
-0.84;
0.84;
0.15;
-0.91;
-0.35;
-0.16;
-0.35;
-1];
f = [0.93;
-1.79;
0.36;
0.52;
0.49;
-1.76;
0.33;
0.48;
0.65;
-1.82;
0.92;
1;
8.8800000000000001;
-2.01;
0.47;
0.49;
0.84;
-2.42;
0.47;
7.15;
0.44;
-3.34;
2.78;
0.44;
0.7;
-6.52;
0.66;
2.32;
1.66;
-1];
w = [10;
10;
10;
10;
10;
10;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
1;
```

```

        1;
        1;
        1;
        1];
lamda = zeros(10,1);
lamda(5) = -0.5;
mu = zeros(8, 1);
point = [int32(3);
        int32(6);
        int32(4);
        int32(5);
        int32(7);
        int32(10);
        int32(8);
        int32(9);
        int32(11);
        int32(13);
        int32(12);
        int32(15);
        int32(14);
        int32(18);
        int32(16);
        int32(17);
        int32(19);
        int32(20);
        int32(21);
        int32(30);
        int32(23);
        int32(26);
        int32(24);
        int32(25);
        int32(27);
        int32(28);
        int32(0);
        int32(29);
        int32(0);
        int32(0);
        int32(2);
        int32(22);
        int32(1);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0);
        int32(0)];
eps = 1e-06;
[lamdaOut, muOut, dl, c, sigma, rank, ifail] = ...
    e02da(x, y, f, w, lamda, mu, point, eps)

lamdaOut =
    -1.0000
    -1.0000
    -1.0000
    -1.0000
    -0.5000
         0
     1.0000
     1.0000
     1.0000

```

```
      1.0000
muOut =
      -1
      -1
      -1
      -1
       1
       1
       1
       1
dl =
      1.0417
      0.0286
      0.0002
      0.0000
      0.0318
      0.0030
      0.0000
      0.0000
      0.0173
      0.0035
      0.0043
      0.0072
      0.0395
      0.0045
      0.0011
      0.0039
      0.0103
      0.0144
      0.0009
      0.0223
      0.0724
      0.0041
      0.0019
      0.0491
c =
      -1.0228
      115.4668
     -433.5558
     -68.1973
      24.8426
    -140.1485
     258.5042
      15.6756
     -29.4878
     132.2933
    -173.5103
      20.0983
       9.9575
     -51.6200
      67.6666
     -5.8765
      10.0577
       4.7543
     -15.3533
     -0.3260
       1.0835
     -2.7932
       7.7708
       0.6315
sigma =
     14.6671
rank =
           22
ifail =
           0
```